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Beamlike (Saint–Venant) solutions for fully anisotropic elastic tubes of arbitrary closed cross section

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Abstract

Beamlike solutions for fully anisotropic elastic tubes of arbitrary closed cross section are derived following the exact beam theory introduced recently by Ladevèze and Simmonds [Comptes Rendus Acad. Sci. Paris 322 (1996) 455; Eur. J. Mech., A/Solids 17 (1998) 377]. Instead of using finite elements to compute the various operators that appear, here the linear shell theory of Koiter [A consistent first approximation in the general theory of thin elastic shells, The Theory of Thin Elastic Shells, Proc. IUTAM Sympos. Delft, Koiter, W.T. (Ed.), North-Holland, Amsterdam, 1959, p. 12] and Sanders [An improved first-approximation theory for thin shells, (1959) NASA Rept. No. 24] is used to exploit the relative thinness of the tube. Analytical, beamlike solutions (the analogues of Saint–Venant solutions in three-dimensional elasticity) are obtained which contain relative errors of $O(h/R)$, where h is the shell thickness and R is some cross sectional radius. These errors are of the same order of magnitude as those contained unavoidably in the stress–strain relations of any first-approximation shell theory. In addition, beamlike stress–strain relations are obtained which express an overall bending strain vector and an overall extensional-shear strain vector in terms of the net traction and moment at any section. Numerical results are presented for tubes with elliptic cross sections. This work generalizes the analysis of Reissner and Tsai [J. Appl. Mech. 39 (1972) 148] by considering external surface loads and by allowing for overall transverse shearing forces in addition to a net axial force and complements the asymptotic analysis of Berdichevsky et al. [Comp. Eng. 2 (1992) 411] by allowing the tube to be of any length.

1. Introduction

Ladevèze and Simmonds (1996, 1998) have shown for a piecewise uniform prismatic body of arbitrary cross section under arbitrary loading that it is possible to construct, from the three-dimensional theory of elasticity, an *exact* beam theory. That is, they present one-dimensional beamlike equations that *involve no approximations whatsoever* (beyond those already embodied in three-dimensional elasticity). In particular,

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there is no assumption that the beam is relatively thin (high aspect ratio) nor are there asymptotic expansions of any kind.

In this exact theory of beams, the three-dimensional solution is decomposed into the sum of a *Saint–Venant* (SV) part and a *residual* or *decaying* (D) part. The SV-part, which is determined first and is, itself, three-dimensional, constitutes not only the sole contribution to the exact one-dimensional beam theory, but also yields a criterion (an orthogonality condition) which determines the D-loads and D-end displacements. These latter (residual) data produce exponentially decaying three-dimensional solutions which contribute nothing to the exact beam theory.

As for the D-part, it is well known in continuum mechanics that when thin layers exist, the governing equations, in the limit as the thickness of the layer approaches zero, may change character completely (e.g., the *elliptic* equations of elasticity become *hyperbolic* in the limit of a saddle-shaped membrane). No mere mathematical curiosity, such limiting behavior can wreak havoc with all-purpose numerical codes designed for bodies of "reasonable" shape. For example, in a solid, elastically isotropic beam of circular cross section, D-solutions in the neighborhoods of ends and/or load and section discontinuities decay over a distance of the order of the radius of the cross section. However, for a relatively thin tube, the D-solutions display *three* decay lengths: one of $O(h)$, the tube thickness (the "three-dimensional" edge effect); another of $O(\sqrt{hR})$, the geometrical mean of the thickness times a characteristic length of the cross section (the "bending" edge effect of classical, first-approximation shell theory); and a third of $O(R\sqrt{R/h})$, the very long "semi-membrane" edge effect. [In elastically orthotropic tubes, the situation is even more complicated—see Sayir (1985) and Simmonds (1992).]

The work of Koiter (1970), Danielson (1970), and Ladevèze (1976, 1980) has shown that, outside a three-dimensional edge zone of width $O(h)$, the linear theory of Sanders (1959) and Koiter (1959) for elastically isotropic or anisotropic shells contains relative mean square errors of $O(h/R + h^2/L^2)$ compared to exact three-dimensional solutions, where L is the "characteristic wavelength" of the deformation pattern.

Now imagine a right cylindrical shell of closed but otherwise arbitrary cross section (i.e., a *tube*) on whose ends certain loads and/or displacements are prescribed. If the end loads (known or not) have a non-zero resultant force or moment, the tube should behave in some overall way like a beam, and, if the tube is thin, this beamlike behavior might be expected to be described to a good approximation by *membrane theory*, provided (near) inextensional deformation is suppressed. Within the framework of the Sanders–Koiter shell theory, we might say, roughly, that as $h \to 0$, membrane theory (the solutions of which yield *overall* beamlike stress–strain relations) corresponds to the SV-solutions of Ladevèze and Simmonds (1996, 1998) for elastic prisms, and that the more complicated supplementary decaying *bending* solutions of the Sanders–Koiter theory correspond to the D-solutions discussed above.

Following the approach in the exact theory of beams, our ultimate goal is take advantage of the relative thinness of the tube to use the Sanders–Koiter theory. Complete analytic solutions can then be derived, as we show. The present paper is devoted to the first step in this program, namely, to develop beamlike (Saint–Venant) solutions for a fully anisotropic tube (21 elastic–geometric constants) of arbitrary (closed) cross section. In particular, we present overall stress–strain relations for a tube subject to surface loads *constant* along its length. Also, we present numerical results for tubes of elliptic cross section.

Of the many studies of thin-walled beams found in the literature, that of Berdichevsky et al. (1992) seems the most relevant. These authors present a commendable review of prior work, including the important paper by Reissner and Tsai (1972), and compare various theories with experiments. As we do, Berdichevsky et al. start with the linear, first-approximation shell theory of Sanders and Koiter applied to an anisotropic tube under surface and end loads. (Initially, they even allow the elastic moduli to vary in the circumferential direction.) Moreover, the early stage of their analysis leads to the same differential equation (29) that we encounter. However, from the outset, Berdichevsky et al. assume that axial variations of the various unknowns are small compared to their circumferential variations and this becomes the basis of their asymptotic analysis in which successive approximations are derived from corresponding forms of the energy functional. In contrast, *we make no approximations beyond those inherent in the Sanders–Koiter theory. Thus, in particular, our analysis applies to tubes of any length whereas that of Berdichevsky et al. is perforce limited to beams with a high length-to-diameter ratio*. Further, as far as practical, we display explicitly the effects of local wall bending stiffness that, although usually small, are accurately predicted by Sanders–Koiter theory. At the same time we note that, although it is possible to find "exact" beamlike solutions of the Sanders–Koiter shell equations, this would lead to totally unnecessary algebraic complications because the constitutive equations of any first-approximation shell theory, as shown by Koiter (1959), contain relative errors of order $\varepsilon = h/R$. We shall exploit this uncertainty to simplify our equations whenever possible.

2. Geometry

In a fixed Euclidean reference frame, let (r, θ, x) denote a set of circular cylindrical coordinates with associated dextral orthonormal base vectors, $\{\mathbf{e}_r(\theta), \mathbf{e}_\theta(\theta), \mathbf{k}\}$. We take

$$\mathscr{T} : \mathbf{x} = R[x\mathbf{k} + \mathbf{r}(y)], \quad \mathbf{r} \in \mathscr{S}, \ 0 \leqslant x \leqslant l, \ 0 \leqslant y \leqslant 2\pi, \tag{1}$$

as the vector representation of the reference surface of the tube. Here, $\mathscr{S}$ is the (suitably scaled) centerline of the cross section of the tube, x and y are, respectively, dimensionless distances along and around $\mathscr{T}$, and $2\pi R$ is the distance around the tube. The dimensionless position of a point on $\mathscr{S}$ may be written as

$$\mathbf{r} = r(y)\mathbf{e}_r(\theta), \tag{2}$$

where

$$\theta = \pm \int_0^y \frac{\sqrt{1 - r^{\bullet 2}(\eta)\,\mathrm{d}\eta}}{r(\eta)}, \tag{3}$$

the ± sign allowing for the possibility that $\mathscr{S}$ might not be star-shaped with respect to the chosen axis of $\mathscr{T}$. (If $\mathscr{T}$ is a circular cylinder of radius R, $r = 1$ and $y = \theta$.) Finally, we shall denote differentiation with respect to x and y by a prime ($'$) and a dot ($^\bullet$), respectively.

When convenient, we shall use Cartesian tensor notation, with $x = x_1$ and $y = x_2$. In particular, if $R^2 a_{\alpha\beta}$ and $Rb_{\alpha\beta}$ denote, respectively, the covariant components of the metric and curvature tensors of $\mathscr{T}$, then, from (1) and (2),

$$a_{\alpha\beta} = \delta_{\alpha\beta}, \qquad b_{\alpha\beta} = \kappa(y)\delta_{\alpha 2}\delta_{\beta 2}, \quad \alpha, \beta = 1, 2, \tag{4}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and

$$\kappa = (r^\bullet \times r^{\bullet\bullet}) \cdot \mathbf{k} = \pm \frac{1 - (1/2)[r^2(y)]^{\bullet\bullet}}{r(y)\sqrt{1 - r^{\bullet 2}(y)}} \tag{5}$$

is the dimensionless curvature of $\mathscr{S}$.

3. The governing equations

Let $\sigma h N_{\alpha\beta}$, $\sigma h^2 M_{\alpha\beta}$, $(\sigma/\overline{E})E_{\alpha\beta}$, $(\sigma/h\overline{E})K_{\alpha\beta}$, and $(\sigma h/R)(p_\alpha, p)$ denote, respectively, the (modified, symmetric) stress resultants, stress couples, extensional strains, bending strains, and surface loads of the Sanders–Koiter theory, where σ is some measure of the stress level in the tube and $\overline{E}$ is some nominal Young's modulus. In component form, with the notation

$$T_{\alpha\beta} = \{T_x, T, T_y\} \tag{6}$$

and with $(\sigma h/R)\{p_x, p_y, -p\}$ denoting the components of the external surface load (assumed known), the equilibrium and compatibility conditions of the Sanders–Koiter theory take the form

$$N'_x + [N - (1/2)\varepsilon\kappa M]^\bullet + p_x = 0 \tag{7}$$

$$N' + N_y^\bullet + \varepsilon\kappa[(3/2)M' + M_y^\bullet] + p_y = 0 \tag{8}$$

$$-\varepsilon(M''_x + 2M'^\bullet + M_y^{\bullet\bullet}) + \kappa N_y = p \tag{9}$$

$$K'_y - [K + (1/2)\varepsilon\kappa E]^\bullet = 0 \tag{7*}$$

$$-K' + K_x^\bullet + \varepsilon\kappa[(3/2)E' - E_x^\bullet] = 0 \tag{8*}$$

$$\varepsilon(E''_y - 2E'^\bullet + E_x^{\bullet\bullet}) + \kappa K_x = 0. \tag{9*}$$

These equations display the *static-geometric duality* of Goldenveiser (1940) and Lure (1940). That is, if we introduce the "hat" notation,

$$\widehat{T}_{\alpha\beta} = e_{\alpha\lambda}e_{\beta\mu}T_{\lambda\mu} = \{T_y, -T, T_x\}, \tag{10}$$

where $e_{\alpha\beta}$ is the two-dimensional alternator, then, on setting $p_x = p_y = p = 0$, the equilibrium equations (7)–(9) go over into the compatibility conditions (7*)–(9*) if the variables below on the left are replaced by those on the right:

$$\widehat{N}_{\alpha\beta} : K_{\alpha\beta}, \quad M_{\alpha\beta} : -\widehat{E}_{\alpha\beta}. \tag{11}$$

To complete the set of field equations, we must add constitutive relations. To exploit fully the economy offered by the static-geometric duality, we follow McDevitt and Simmonds (1999) and write these in the form

$$-\widehat{E}_{\alpha\beta} = \psi_{,\widehat{N}_{\alpha\beta}} = -A_{\alpha\beta\lambda\mu}\widehat{N}_{\lambda\mu} + C_{\alpha\beta\lambda\mu}K_{\lambda\mu} \tag{12}$$

$$M_{\alpha\beta} = \psi_{,K_{\alpha\beta}} = A^*_{\alpha\beta\lambda\mu}K_{\lambda\mu} + C^*_{\alpha\beta\lambda\mu}\widehat{N}_{\lambda\mu}, \tag{12*}$$

where $A^*_{\alpha\beta\lambda\mu}$ is the dual of $-A_{\alpha\beta\lambda\mu}$, $C^*_{\alpha\beta\lambda\mu} \equiv C_{\lambda\mu\alpha\beta}$ is the dual of $C_{\alpha\beta\lambda\mu}$, and

$$\psi = (1/2)(A^*_{\alpha\beta\lambda\mu}K_{\alpha\beta}K_{\lambda\mu} + C_{\alpha\beta\lambda\mu}\widehat{N}_{\alpha\beta}K_{\lambda\mu} + C^*_{\alpha\beta\lambda\mu}K_{\alpha\beta}\widehat{N}_{\lambda\mu} - A_{\alpha\beta\lambda\mu}\widehat{N}_{\alpha\beta}\widehat{N}_{\lambda\mu}) \tag{13}$$

is the dimensionless mixed-energy density. In (13), no type of elastic symmetry is assumed (as is sometimes imposed for simplicity on a *solid* anisotropic cylinder). Rather, one should regard (13) as the mixed-energy density of a thin, totally anisotropic sheet bent to form a tube of arbitrary cross section.

4. The tube as a beam

Let

$$\mathbf{p} = p_x\mathbf{k} + p_y\mathbf{t} - p\mathbf{n} = \mathbf{p}(x, y) \tag{14}$$

and

$$\mathbf{N}_x = N_x\mathbf{k} + [N + (3/2)\varepsilon\kappa M]\mathbf{t} - \varepsilon(M'_x + 2M^\bullet)\mathbf{n} = N_x\mathbf{k} + G\mathbf{t} - \varepsilon[M'_x\mathbf{n} + 2(M\mathbf{n})^\bullet] = \mathbf{N}_x(x, y; \varepsilon), \tag{15}$$

where

$$G = N - (1/2)\varepsilon\kappa M. \tag{16}$$

In (14) and (15)

$$\mathbf{t} = \mathbf{r}^{\bullet}(\mathbf{y}) \quad \text{and} \quad \mathbf{n} = \mathbf{k} \times \mathbf{t}(y) \tag{17}$$

are, respectively, a unit tangent and a unit inward normal to $\mathscr{S}$. By computing from the local equilibrium equations (7)–(9), at a dimensionless distance x from the left end of the tube, the net *traction* $\sigma hR\mathbf{T}$ and the net *moment* $\sigma hR^2\mathbf{M}$ about the *center* ($r = 0$) of $\mathscr{S}$, we obtain the following beamlike equilibrium equations:

$$\mathbf{T}' + \mathbf{P} = \mathbf{0} \tag{18}$$

and

$$\mathbf{M}' + \mathbf{k} \times \mathbf{T} + \mathbf{L} = \mathbf{0}, \tag{19}$$

where

$$\mathbf{T} = \int_0^{2\pi} \mathbf{N}_x(x, y; \varepsilon)\,\mathrm{d}y \tag{20}$$

$$\mathbf{P} = \int_0^{2\pi} \mathbf{p}(x, y)\,\mathrm{d}y \tag{21}$$

$$\mathbf{M} = \int_0^{2\pi} [\mathbf{r}(y) \times \mathbf{N}_x(x, y; \varepsilon) - \varepsilon\mathbf{M}_x(x, y; \varepsilon)\mathbf{t}(y)]\,\mathrm{d}y \tag{22}$$

$$\mathbf{L} = \int_0^{2\pi} \mathbf{r}(y) \times \mathbf{p}(x, y)\,\mathrm{d}y. \tag{23}$$

We note that (18)–(23) are *exact* consequences of the linear equations of equilibrium of three-dimensional continuum mechanics (Budiansky and Sanders, 1963). We also note that the definitions of $\mathbf{T}$ and $\mathbf{M}$ are independent of any material properties.

Given $\mathbf{T}(0)$ and $\mathbf{M}(0)$ and the surface load $\mathbf{p}(x, y)$, the solution of the beam equations is immediate:

$$\mathbf{T} = \mathbf{T}(0) - \int_0^x \mathbf{P}(\xi)\,\mathrm{d}(\xi) \tag{24}$$

$$\mathbf{M} = \mathbf{M}(0) - x\mathbf{k} \times \mathbf{T}(x) - \int_0^x [\xi\mathbf{k} \times \mathbf{P}(\xi) + \mathbf{L}(\xi)]\,\mathrm{d}\xi \tag{25}$$

We will return to the beam equations once we have identified those particular solutions of our field equations (7)–(9*), (12) and (12*) that are *beamlike*.

We note that the static-geometric duality (11) implies immediately that the beam equations (18) and (19) have the kinematic duals

$$\mathbf{A}' = \mathbf{0} \tag{18*}$$

$$\mathbf{B}' + \mathbf{k} \times \mathbf{A} = \mathbf{0} \tag{19*}$$

where, in analogy with (20) and (22),

$$\mathbf{A} = \int_0^{2\pi} \mathbf{K}_y(x, y; \varepsilon)\,\mathrm{d}y \tag{20*}$$

$$\mathbf{B} = \int_0^{2\pi} [\mathbf{r}(y) \times \mathbf{K}_y(x,y;\varepsilon) + \varepsilon E_y(x,y;\varepsilon)\mathbf{t}(y)]\,\mathrm{d}y. \tag{22*}$$

Here, by analogy with (15),

$$\mathbf{K}_y = K_y\mathbf{k} - [K - (3/2)\varepsilon\kappa E]\mathbf{t} + \varepsilon(E'_y - 2E^\bullet)\mathbf{n} = K_y\mathbf{k} - H\mathbf{t} + \varepsilon[E'_y\mathbf{n} - 2(E\mathbf{n})^\bullet] \equiv \mathbf{K}_y(x,y;\varepsilon) \tag{15*}$$

where, in analogy with (16),

$$H = K + (1/2)\varepsilon\kappa E. \tag{16*}$$

Let $(R^2/h)(\sigma/\overline{E})(\mathbf{U} + \varepsilon\mathbf{u})$ denote the shell displacement, where

$$\begin{aligned}\mathbf{U} + \varepsilon\mathbf{u} = {} & U(y)\mathbf{k} + [V(y) - xU^\bullet(y)]\mathbf{t}(y) + \rho(y)[V(y) - xU^\bullet(y)]^\bullet\mathbf{n}(y) \\ & + \varepsilon[u(x,y;\varepsilon)\mathbf{k} + v(x,y;\varepsilon)\mathbf{t}(y) - w(x,y;\varepsilon)\mathbf{n}(y)].\end{aligned} \tag{26}$$

Here, $\mathbf{U}$ and $\mathbf{u}$ represent, respectively, an *inextensional* and a *residual* displacement, and $\rho = \kappa^{-1}$. If we introduce the (dimensionless) strain–displacement relations of the Sanders–Koiter theory, namely,

$$E_x = u', \quad E = (1/2)(u^\bullet + v'), \quad E_y = v^\bullet + \kappa w \tag{27}$$

and

$$\begin{aligned}& K_x = -\varepsilon w'', \quad K = -\mathscr{M}(U^\bullet) - \varepsilon[w'^\bullet - (3/4)\kappa v' + (1/4)\kappa u^\bullet] \\ & K_y = [\mathscr{M}(V - xU^\bullet)]^\bullet - \varepsilon(w^\bullet - \kappa v)^\bullet,\end{aligned} \tag{28}$$

where

$$\mathscr{M} = \frac{\mathrm{d}}{\mathrm{d}y}\left[\rho(y)\frac{\mathrm{d}}{\mathrm{d}y}\right] + \kappa(y), \tag{29}$$

then from (15*), (28) and (29),

$$\mathbf{K}_y = \{[\mathscr{M}(V - xU^\bullet) - \varepsilon(w^\bullet - \kappa v)]\mathbf{k} + (\rho U^{\bullet\bullet} + \varepsilon w')\mathbf{t} - (U^\bullet + \varepsilon u^\bullet)\mathbf{n}\}^\bullet. \tag{30}$$

On using $(28)_3$ and (30) in (20*) and (22*), we see that $\mathbf{A}$ and $\mathbf{B}$ represent gross dislocations of the closed cross section $\mathscr{S}$ that vanish if $\mathscr{S}$ is neither cut, nor cut and then welded to produce a dislocation, as we now assume. That is, we assume $\mathbf{U} + \varepsilon\mathbf{u}$ to be 2π-periodic in y.

5. Beamlike solutions

We now assume that the (dimensionless) external loads are independent of the axial coordinate x, which we indicate by writing

$$\mathbf{p} = \overset{0}{p}_x(y)\mathbf{k} + \overset{0}{p}_y(y)\mathbf{t}(y) - \overset{0}{p}(y)\mathbf{n}(y) = \overset{0}{p}_x(y)\mathbf{k} + \overset{0}{q}(y)\mathbf{t}(y) - [\rho(y)\overset{0}{p}(y)\mathbf{t}(y)]^\bullet, \tag{31}$$

where

$$\overset{0}{q} = [\rho(y)\overset{0}{p}(y)]^\bullet + \overset{0}{p}_y(y). \tag{32}$$

Consistent with (31), we look for solutions of our field equations (7)–(9*), 12 and 12* that are *quadratic in x*. To this end we introduce the notation

$$f(x,y) = \overset{0}{f}(y) + x\overset{1}{f}(y) + x^2\overset{2}{f}(y). \tag{33}$$

Anticipating the order of magnitude of certain terms, we introduce the following representations for the stress resultants and bending strains:

$$N_x = \overset{0}{N}_x(y;\varepsilon) - x[\overset{0}{G}{}^{\bullet}(y;\varepsilon) + \overset{0}{p}_x(y)] + (x^2/2)[\overset{0}{q}(y) + \varepsilon\overset{1}{\mathscr{G}}(y;\varepsilon)]^{\bullet} \tag{34}$$

$$G = \overset{0}{G}(y;\varepsilon) - x\overset{0}{q}(y) - \varepsilon x\overset{1}{\mathscr{G}}(y;\varepsilon) \tag{35}$$

$$N_y = \rho(y)\overset{0}{p}(y) + \varepsilon[\overset{0}{\mathscr{N}}_y(y;\varepsilon) + x\overset{1}{\mathscr{N}}_y(y;\varepsilon) + x^2\overset{2}{\mathscr{N}}_y(y;\varepsilon)] \tag{36}$$

$$K_y = \overset{0}{K}_y(y;\varepsilon) + x\overset{0}{H}{}^{\bullet}(y;\varepsilon) - \varepsilon(x^2/2)\overset{1}{\mathscr{H}}{}^{\bullet}(y;\varepsilon) \tag{34*}$$

$$H = \overset{0}{H}(y;\varepsilon) - \varepsilon x\overset{1}{\mathscr{H}}(y;\varepsilon) \tag{35*}$$

$$K_x = \varepsilon[\overset{0}{\mathscr{K}}_x(y;\varepsilon) + x\overset{1}{\mathscr{K}}_x(y;\varepsilon) + x^2\overset{2}{\mathscr{K}}_x(y;\varepsilon)], \tag{36*}$$

where $\overset{0}{N}_x$, $\overset{0}{G}$, $\overset{0}{H}$, $\overset{1}{\mathscr{G}}$, $\overset{0}{\mathscr{N}}_y$, $\overset{1}{\mathscr{N}}_y$, $\overset{2}{\mathscr{N}}_y$, $\overset{0}{K}_y$, $\overset{1}{\mathscr{H}}$, $\overset{0}{\mathscr{K}}_x$, $\overset{1}{\mathscr{K}}_x$, and $\overset{2}{\mathscr{K}}_x$ are new unknowns that depend only on the dimensionless circumferential coordinate y. These expressions satisfy (7) and (7*) identically. From (24) and (25), we see that all terms quadratic in x come from the force term $\mathbf{k} \times \mathbf{P}$.

Turning to (8) and (9) and inserting (35) and (36), we obtain, on equating coefficients of x^2,

$$\overset{2}{\mathscr{N}}{}^{\bullet}_y + \kappa\overset{2}{M}{}^{\bullet}_y = 0 \tag{8$_2$}$$

$$-\overset{2}{M}{}^{\bullet\bullet}_y + \kappa\overset{2}{\mathscr{N}}_y = 0. \tag{9$_2$}$$

Eliminating $\overset{2}{\mathscr{N}}_y$ by differentiation, we see that $\overset{2}{M}{}^{\bullet}_y$ must satisfy the second-order differential equation

$$\mathscr{M}(\overset{2}{M}{}^{\bullet}_y) = 0 \tag{37}$$

where $\mathscr{M}$ is the differential operator defined by (29). Since (5) and (17) imply that

$$\mathbf{t}^{\bullet} = \kappa(y)\mathbf{n}(y) \quad \text{and} \quad \mathbf{n}^{\bullet} = -\kappa(y)\mathbf{t}(y), \tag{38}$$

it may be verified that the general homogeneous solution of (37) is

$$\overset{2}{M}_y = \overset{2}{\alpha} + \overset{2}{\boldsymbol{\alpha}} \cdot \mathbf{r}(y), \tag{39}$$

where $\overset{2}{\alpha}$ and $\overset{2}{\boldsymbol{\alpha}}$ are, respectively, an unknown constant scalar and vector. (Here and henceforth, we adopt the convention that any constants that depend, ultimately, on the surface loads only, are denoted by Greek letters.) Substituting (39) back into ((9)$_2$) and canceling a common factor of κ, we obtain

$$\overset{2}{\mathscr{N}}_y = \overset{2}{\boldsymbol{\alpha}} \cdot \mathbf{n}(y). \tag{40}$$

Next, again using (35) and (36), we equate coefficients of x^1 in (8) and (9) to obtain

$$\overset{1}{\mathscr{N}}{}^{\bullet}_y + \kappa(4\overset{2}{M} + \overset{1}{M}{}^{\bullet}_y) = 0 \tag{8$_1$}$$

$$-(4\overset{2}{M} + \overset{1}{M}{}^{\bullet}_y)^{\bullet} + \kappa\overset{1}{\mathscr{N}}_y = 0. \tag{9$_1$}$$

Eliminating $\overset{1}{\mathscr{N}}_y$ by differentiation, we obtain an equation of the same form as (37), whose general solution is

$$4\overset{2}{M} + \overset{1}{M}{}^{\bullet}{}_y = \overset{1}{\mathbf{a}} \cdot \mathbf{t}(y), \tag{41}$$

where $\overset{1}{\mathbf{a}}$ is an unknown constant vector.

Substituting (41) back into $((9)_1)$ and canceling a common factor of κ, we have

$$\overset{1}{\mathscr{N}}_y = \overset{1}{\mathbf{a}} \cdot \mathbf{n}(y). \tag{42}$$

Finally, we equate the coefficients of x^0 obtained by substituting (35) and (36) into (8) and (9). This gives

$$-\overset{1}{\mathscr{G}} + \overset{0}{\mathscr{N}}{}^{\bullet}_y + \kappa(2\overset{1}{M} + \overset{0}{M}{}^{\bullet}_y) = 0 \qquad (8)_0$$

$$-[2\overset{2}{M}_x + (2\overset{1}{M} + \overset{0}{M}{}^{\bullet}_y)^{\bullet}] + \kappa\overset{0}{\mathscr{N}}_y = 0. \qquad (9)_0$$

From this last equation,

$$\overset{0}{\mathscr{N}}_y = \rho[2\overset{2}{M}_x + (2\overset{1}{M} + \overset{0}{M}{}^{\bullet}_y)^{\bullet}] \tag{43}$$

and substitution of this expression into $(8)_0$ yields

$$\overset{1}{\mathscr{G}} = 2(\rho\overset{2}{M}_x)^{\bullet} + \mathscr{M}(2\overset{1}{M} + \overset{0}{M}{}^{\bullet}_y). \tag{44}$$

By the static-geometric duality, the analogous kinematic solutions and relations for (8^*) and (9^*) are

$$-\overset{2}{E}_x = \overset{2}{\alpha}{}^* + \overset{2}{\boldsymbol{\alpha}}{}^* \cdot \mathbf{r}(y) \qquad (39^*)$$

$$\overset{2}{\mathscr{K}}_x = \overset{2}{\boldsymbol{\alpha}}{}^* \cdot \mathbf{n}(y) \qquad (40^*)$$

$$4\overset{2}{E} - \overset{1}{E}{}^{\bullet}_x = \overset{1}{\mathbf{a}}{}^* \cdot \mathbf{t}(y) \qquad (41^*)$$

$$\overset{1}{\mathscr{K}}_x = \overset{1}{\mathbf{a}}{}^* \cdot \mathbf{n}(y) \qquad (42^*)$$

$$\overset{0}{\mathscr{K}}_x = \rho[-2\overset{2}{E}_y + (2\overset{1}{E} - \overset{0}{E}{}^{\bullet}_x)^{\bullet}], \qquad (43^*)$$

$$-\overset{1}{\mathscr{H}} = -2(\rho\overset{2}{E}_y)^{\bullet} + \mathscr{M}(2\overset{1}{E} - \overset{0}{E}{}^{\bullet}_x) \qquad (44^*)$$

where $\overset{2}{\alpha}{}^*$, $\overset{2}{\boldsymbol{\alpha}}{}^*$, etc., are constant scalars and vectors.

Note that we have satisfied *exactly* both the equilibrium and compatibility conditions *without* introducing stress–strain relations. We now introduce the latter to infer consistent forms for $\overset{0}{G}$, $\overset{0}{H}$, and the external loads $\overset{0}{p}_x$ and $\overset{0}{q}$. With these in hand, we can return to the beam equations to determine all the remaining unknowns, which, at this point, are $\overset{0}{N}_x$, $\overset{0}{G}$, $\overset{0}{K}_y$, $\overset{0}{H}$, $\overset{0}{\mathscr{N}}_y$, $\overset{0}{\mathscr{K}}_x$, and the constants $\overset{2}{\alpha}$, $\overset{2}{\alpha}{}^*$, $\overset{2}{\boldsymbol{\alpha}}$, $\overset{2}{\boldsymbol{\alpha}}{}^*$, $\overset{1}{\mathbf{a}}$, and $\overset{1}{\mathbf{a}}{}^*$.

First, inserting (16), (34), (34^*), (36) and (36^*) into the expression for $\overset{2}{M}_y$ coming from (12^*), we find that (39) takes the form

$$(1/2)C^{*}_{2222}\overset{0}{q}{}^{\bullet} + O(\varepsilon) = \overset{2}{\alpha} + \overset{2}{\boldsymbol{\alpha}} \cdot \mathbf{r}(y). \tag{45}$$

To simplify the formulas to follow, we now take the center of $\mathscr{S}$ to coincide with its *centroid*, which means that we take

$$\int_0^{2\pi} \mathbf{r}(y)\,\mathrm{d}y = \mathbf{0}. \tag{46}$$

Furthermore, because the stress–strain relations of any first-approximation shell theory contain unavoidable relative errors of $O(\varepsilon)$, as shown by Koiter (1959), we take advantage of this by subtracting from the relation for $\overset{2}{M}_y$ those $O(\varepsilon)$-terms that appear in (45). In general, such a minor modification yields new stress–strain relations which are *not* derivable exactly from a strain–energy density. However, Simmonds (1971) and Koiter and Clément (1979) have shown that such modifications lead to relative mean square errors of only $O(\varepsilon)$ and we shall make them whenever they lead to simplified formulas. This observation leads to the general (and useful) conclusion that, in first-approximation shell theory, corrections of relative order ε are meaningful for stress resultants and bending strains, but not for stress couples and extensional strains.

If we now integrate both sides of (45) from 0 to 2π, (46) implies that $\overset{2}{\alpha} = 0$. This, in turn, means that $\overset{0}{q}$, which, by hypothesis, is independent of x and ε, must be of the form

$$\overset{0}{q} = \alpha - \boldsymbol{\alpha} \cdot \mathbf{s}(y), \tag{47}$$

where α and $\boldsymbol{\alpha}$ are, respectively, new unknown constants and

$$\mathbf{s} \equiv \int_0^{y} \mathbf{r}(\eta)\,\mathrm{d}\eta + (1/2\pi)\int_0^{2\pi} y\mathbf{r}(y)\,\mathrm{d}y. \tag{48}$$

[If the elastic coefficients are allowed to depend on y, as in Reissner and Tsai (1972), then $\overset{0}{q}$ still has the same form as (47), but the function $\mathbf{s}(y)$ becomes more complicated since it now depends on $C^{*}_{2222}(y)$.] By (46), $\mathbf{s}(2\pi) = \mathbf{s}(0)$. Furthermore, the last (constant) term in (48) has been chosen so that

$$\int_0^{2\pi} \mathbf{s}(y)\,\mathrm{d}y = \int_0^{2\pi}\int_0^{y} \mathbf{r}(\eta)\,\mathrm{d}\eta + \int_0^{2\pi} y\mathbf{r}(y)\,\mathrm{d}y = 2\pi\int_0^{2\pi} \mathbf{r}(y)\,\mathrm{d}y = \mathbf{0} \tag{49}$$

[If $\mathscr{S}$ is a circle, $\mathbf{s} = -\mathbf{e}_\theta(\theta)$.] Henceforth, we regard $\overset{0}{q}$ as known—see $(60)_2$ and $(64)_1$—so that (45) yields $\overset{2}{\boldsymbol{\alpha}} = -(1/2)C^{*}_{2222}\boldsymbol{\alpha}$. Note from (12), (34)–(36*), and (39*) that the static–geometric dual of (45) is

$$-(1/2)A_{2222}\overset{0}{q}{}^{\bullet} = \overset{2}{\boldsymbol{\alpha}}{}^{*} \cdot \mathbf{r}(y)\overset{2}{\alpha}{}^{*} = 0, \tag{45*}$$

where, in analogy with what we did to simplify (45), we have added certain $O(\varepsilon)$-terms to the stress–strain relation for $\overset{2}{E}_x$. Thus, even if bending and stretching are uncoupled in the stress–strain relations ($C_{\alpha\beta\lambda\mu} = 0$), $\overset{0}{q}$ must have the form (47) to satisfy (45*), which yields $\overset{2}{\boldsymbol{\alpha}}{}^{*} = (1/2)A_{2222}\boldsymbol{\alpha}$. Note also that the static-geometric duality does *not* extend to load terms so that, for example, $-A_{2222}$ in (45*) is not the dual of C^{*}_{2222} in (45).

At this point we note that the nominal Young's modulus, $\overline{E}$, introduced as a non-dimensionalizing factor at the beginning of Section 3, may always be chosen so that $A_{2222} = 1$ which we shall do henceforth. That is, we now choose $\overline{E}$ to be the longitudinal Young's modulus.

Next, we insert (12*) and (34)–(36*), into (41) and modify the stress–strain relation for $\overset{2}{M}$ and $\overset{1}{M}_y$ by adding certain $O(\varepsilon)$-terms. Integrating once, we obtain

$$A^{*}_{2222}\overset{0}{H}{}^{\bullet} - C^{*}_{2222}(\overset{0}{G}{}^{\bullet} + \overset{0}{p}_x) + 4\overline{C}^{*}_{\underline{1222}}\overset{0}{q} = \overset{1}{\alpha} + \overset{1}{\mathbf{a}} \cdot \mathbf{r}(y), \tag{50}$$

where $2\overline{C}^*_{1222} \equiv C^*_{1222} + C^*_{2212}$ and $\overset{1}{\alpha}$ is an unknown constant. From 12, (34)–(36*) and (41*), the static-geometric dual of this equation is

$$\overset{0}{G}{}^\bullet + \overset{0}{p}_x + C_{2222}\overset{0}{H}{}^\bullet - 4A_{\underline{1222}}\overset{0}{q} = \overset{1}{\alpha}{}^* + \overset{1}{\mathbf{a}}{}^* \cdot \mathbf{r}(y), \tag{50*}$$

where the symmetry $A_{1222} = A_{2212}$ has been used. As shall see, many of the subsequent formulas simplify considerably if the underlined terms in (50) and (50*) are absent. We call this *extended orthotropy* and note that such a simplification was introduced at a certain stage by Ladevèze and Simmonds (1998) in their treatment of arbitrary prismatic beams. However, here we prefer to retain these coupling terms as far as practical to explore the implications of full anisotropy.

Because (50) and (50*) must hold for all possible values of the elastic constants (consistent with a positive-definite strain–energy density), because $\overset{0}{p}_x$, $\overset{0}{q}$, and $\overset{0}{G}{}^\bullet$ may be prescribed independently (as we shall see), and because $\int_0^{2\pi}(\overset{0}{G}, \overset{0}{H})^\bullet \,\mathrm{d}y = (0, 0)$, the quantities $\overset{0}{p}_x$, $\overset{0}{G}{}^\bullet$, and $\overset{0}{H}{}^\bullet$ must have the forms

$$\overset{0}{p}_x = \beta + \boldsymbol{\beta} \cdot \mathbf{r}(y), \tag{51}$$

$$\overset{0}{G}{}^\bullet = -\mathbf{b} \cdot \mathbf{r}(y) + \underline{\boldsymbol{\gamma}} \cdot \mathbf{s}(y), \tag{52}$$ [2]

$$\underline{\overset{0}{H}{}^\bullet} = -\underline{\boldsymbol{\lambda}}^* \cdot \mathbf{r}(y) + \underline{\boldsymbol{\gamma}}^* \cdot \mathbf{s}(y), \tag{52*}$$

where $\beta, \boldsymbol{\beta}, \ldots, \boldsymbol{\gamma}^*$ are unknown constant scalars and vectors. We satisfy (50) and (50*) by taking

$$\{\underline{\boldsymbol{\gamma}}, \underline{\boldsymbol{\gamma}}^*\} = -\frac{4\{A_{1222}A^*_{2222} + C_{2222}\overline{C}^*_{1222}, A_{1222}C^*_{2222} - \overline{C}^*_{1222}\}}{A^*_{2222} + C_{2222}C^*_{2222}}\boldsymbol{\alpha} \equiv \{\underline{\Gamma}, \underline{\Gamma}^*\}\boldsymbol{\alpha} \tag{53}$$

and expressing $\overset{1}{\alpha}$, $\overset{1}{\alpha}{}^*$, $\overset{1}{\mathbf{a}}$, and $\overset{1}{\mathbf{a}}{}^*$ in terms of β, $\boldsymbol{\beta} - \mathbf{b}$, and $\mathbf{b}^*$. Note that if bending and stretching are uncoupled in the stress–strain relations, $\underline{\boldsymbol{\gamma}} = -4A_{1222}\boldsymbol{\alpha}$ and $\underline{\boldsymbol{\gamma}}^* = \mathbf{0}$.

Integrating (52) and (52*), using (48), and defining

$$\mathbf{v} = \int_0^y \mathbf{s}(\eta)\,\mathrm{d}\eta + (1/2\pi)\int_0^{2\pi} y\mathbf{s}(y)\,\mathrm{d}y, \tag{54}$$ [3]

so that

$$\mathbf{v}(2\pi) = \mathbf{v}(0) \quad \text{and} \quad \int_0^{2\pi} \mathbf{v}(y)\,\mathrm{d}y = \mathbf{0}, \tag{55}$$

we have

$$\overset{0}{G} = b - \mathbf{b} \cdot \mathbf{s}(y) + \underline{\boldsymbol{\gamma}} \cdot \mathbf{v}(y) \tag{56}$$

and

$$\underline{\overset{0}{H}} = \underline{\lambda}^* - \underline{\boldsymbol{\lambda}}^* \cdot \mathbf{s}(y) + \underline{\boldsymbol{\gamma}}^* \cdot \mathbf{v}(y), \tag{56*}$$

where the constant terms coming from (48) and (54) have been absorbed into the unknown constants b and $\underline{\lambda}^*$.

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[2] The term $\underline{\boldsymbol{\gamma}}\cdot\mathbf{s}$ could be partitioned between (51) and (52). However, we prefer to keep the expression for $\overset{0}{p}_x$ as simple as possible.

[3] If $\mathscr{S}$ is a circle, $\mathbf{v} = -\mathbf{e}_r(\theta)$.

6. Determination of the constants in the beamlike solutions

For what follows, it is useful to note by $(17)_1$ that

$$\int_0^{2\pi} \mathbf{t}(y)\,\mathrm{d}y = \mathbf{0}. \tag{57}$$

Inserting (47) and (51) into (31) and the resulting expression into (21), we have, by (46), (49) and (57),

$$\mathbf{P} = \pi(2\beta\mathbf{k} + \boldsymbol{\alpha}\cdot\mathbf{I}) \tag{58}$$

where

$$\mathbf{I}(y) \equiv -(1/\pi)\int_0^y \mathbf{s}(\eta)\mathbf{t}(\eta)\,\mathrm{d}\eta = (1/\pi)\int_0^y \mathbf{r}(\eta)\mathbf{r}(\eta)\,\mathrm{d}\eta = \mathbf{I}^{\mathrm{T}}(y), \quad \mathbf{I} = \mathbf{I}(2\pi), \tag{59}$$

is a two dimensional, second-order tensor in the plane of $\mathscr{S}$. [Here, we have denoted the dyadic product of two vectors **a** and **b** by **ab**. Further, for later use, we have defined a variable tensor, $\mathbf{I}(y)$, even though we need its value only at $y = 2\pi$ in (58). Note that if $\mathscr{S}$ is a circle, $\mathbf{I} = \mathbf{1}$, where **1** is the two-dimensional identity tensor.] Thus,

$$\beta = (1/2\pi)\mathbf{P}\cdot\mathbf{k} \quad \text{and} \quad \boldsymbol{\alpha} = (1/\pi)\mathbf{P}\cdot\mathbf{I}^{-1}. \tag{60}$$

Turning to (23) and noting (31), (46), (47), (51) and (59), we have

$$\mathbf{L} = (2\alpha A + \pi\boldsymbol{\alpha}\cdot\mathbf{m})\mathbf{k} + \pi\boldsymbol{\beta}\cdot\mathbf{I}\times\mathbf{k}. \tag{61}$$

By the divergence theorem,

$$A(y) = (1/2)\mathbf{k}\cdot\int_0^y \mathbf{r}(\eta)\times\mathbf{t}(\eta)\,\mathrm{d}\eta = -(1/2)\int_0^y \mathbf{r}(\eta)\cdot\mathbf{n}(\eta)\,\mathrm{d}\eta, \quad A = A(2\pi), \tag{62}$$

is the area swept out by the vector $\mathbf{r}(\eta)$ as η goes from 0 to y and

$$\mathbf{m}(y) \equiv (1/\pi)\int_0^y \mathbf{s}(\eta)\mathbf{r}(\eta)\cdot\mathbf{n}(\eta)\,\mathrm{d}\eta, \quad \mathbf{m} = \mathbf{m}(2\pi). \tag{63}$$

(If $\mathscr{S}$ is a circle, $A = B = \pi$ and $\mathbf{m} = \mathbf{0}$.) Thus, with the use of $(60)_2$,

$$\alpha = (1/2A)(\mathbf{L}\cdot\mathbf{k} - \mathbf{P}\cdot\mathbf{I}^{-1}\cdot\mathbf{m}) \quad \text{and} \quad \boldsymbol{\beta} = (1/\pi)\mathbf{I}^{-1}\cdot(\mathbf{k}\times\mathbf{L}). \tag{64}$$

[In terms of the given surface loads, $\alpha = \int_0^{2\pi} \overset{0}{p}_y(y)\,\mathrm{d}y$ and $\boldsymbol{\beta} = (l/\pi)\mathbf{I}^{-1}\int_0^{2\pi} \overset{0}{p}_x(y)\mathbf{r}(y)\,\mathrm{d}y$.]

Note that the (given) distributed beam loads **P** and **L** determine $\overset{0}{p}_x$ and $\overset{0}{q}$, but not $\overset{0}{p}$. Hence, *as part of our definition of a beamlike solution, we simply set* $\overset{0}{p} = 0$.

Next, setting $x = 0$ in (15), (20), (34) and (35), we have

$$\mathbf{T}(0) = \int_0^{2\pi} [\overset{0}{N}_x(y;\varepsilon)\mathbf{k} + \overset{0}{G}(y;\varepsilon)\mathbf{t}(y) - \varepsilon\overset{1}{M}_x(y;\varepsilon)\mathbf{n}(y)]\,\mathrm{d}y. \tag{65}$$

So far, we have put no restrictions on the form of $\overset{0}{N}_x$ or its static-geometric dual, $\overset{0}{K}_y$. However, it proves to be sufficient (and may be taken as part of the definition of a beamlike solution) to take

$$\overset{0}{N}_x = d + \mathbf{d}\cdot\mathbf{r}(y), \quad \overset{0}{K}_y = \mathbf{d}^*\cdot\mathbf{r}(y), \tag{66, 66*}$$

where d, $\mathbf{d}$, and $\mathbf{d}^*$ are unknown constants; $d^* = 0$ because $\int_0^{2\pi} K_y\,\mathrm{d}y = 0$. [Reissner and Tsai (1972) make an assumption similar to (66) and (66*).] By (12*), (34)–(36), (46), (47), (51)–(52*), (56), (56)*, (57), (59), (62), (66) and (66*),

$$\mathbf{T}(0) = 2\pi d\mathbf{k} + \pi\mathbf{b}\cdot\mathbf{I} + B\underline{\boldsymbol{\gamma}}\times\mathbf{k} - \varepsilon\Big\{2A[A^*_{1122}\underline{\boldsymbol{\lambda}^*} + C^*_{1122}(\boldsymbol{\beta}-\mathbf{b})] + \pi(A^*_{1122}\underline{\boldsymbol{\gamma}^*} - 2\underline{C^*_{1112}}\boldsymbol{\alpha} - C^*_{1122}\underline{\boldsymbol{\gamma}})\cdot\mathbf{I}\times\mathbf{k}\Big\} \tag{67}$$

where

$$B(y) = (1/2)\mathbf{k}\cdot\int_0^y \mathbf{s}(\eta)\times\mathbf{r}(\eta)\,\mathrm{d}\eta, \quad B = B(2\pi), \tag{68}$$

is the area swept out by $\mathbf{s}(\eta)$ as η goes from 0 to y.

Finally, from (15) and (22),

$$\mathbf{M}(0) = \int_0^{2\pi}\{\mathbf{r}(y)\times[\overset{0}{N}_x(y;\varepsilon)\mathbf{k} + \overset{0}{G}(y;\varepsilon)\mathbf{t}(y)] + \varepsilon[2\overset{0}{M}(y;\varepsilon)\mathbf{k} - \overset{0}{M}_x(y;\varepsilon)\mathbf{t}(y) - \overset{1}{M}_x(y;\varepsilon)\mathbf{r}(y)\times\mathbf{n}(y)]\}\,\mathrm{d}y \tag{69}$$

or, by of the same equations that led to (67),

$$\begin{aligned}\mathbf{M}(0) = {} & [2Ab + \pi(\mathbf{b}\cdot\mathbf{m} - \underline{\boldsymbol{\gamma}}\cdot\mathbf{q})]\mathbf{k} - \pi\mathbf{d}\cdot\mathbf{I}\times\mathbf{k} + \varepsilon\{4\pi(\underline{C^*_{1222}}d - 2C^*_{1212}b) + (A^*_{1122}\underline{\boldsymbol{\gamma}^*} - 2\underline{C^*_{1122}}\boldsymbol{\alpha} - C^*_{1122}\underline{\boldsymbol{\gamma}})\cdot\mathbf{n_r} \\ & - [A^*_{1122}\underline{\boldsymbol{\lambda}} + C^*_{1122}(\boldsymbol{\beta}-\mathbf{b})]\cdot\mathbf{n_t}\}\mathbf{k} + \varepsilon[2C^*_{1212}(\pi\mathbf{b}\cdot\mathbf{I} - B\underline{\boldsymbol{\gamma}}\times\mathbf{k}) - 2\underline{A^*_{1222}}(\pi\underline{\boldsymbol{\lambda}}^*\cdot\mathbf{I} - B\underline{\boldsymbol{\gamma}}^*\times\mathbf{k}) \\ & - AC^*_{1122}\mathbf{d}\times\mathbf{k}],\end{aligned} \tag{70}$$

where

$$\mathbf{q}(y) \equiv (1/\pi)\int_0^y \mathbf{v}(\eta)\mathbf{r}(\eta)\cdot\mathbf{n}(\eta)\,\mathrm{d}\eta, \quad \mathbf{q} = \mathbf{q}(2\pi) \tag{71}$$

and

$$\mathbf{n}_{\{\mathbf{r},\mathbf{t}\}} \equiv (1/2)\int_0^{2\pi} r^2(\eta)\{\mathbf{r}(\eta),\mathbf{t}(\eta)\}\mathrm{d}(\eta). \tag{72}$$

(For a circle, $\mathbf{q} = \mathbf{n_r} = \mathbf{n_t} = \mathbf{0}$.)

Remarks

(1) Only seven of the 20 independent elastic constants—recall that we have set $A_{2222} = 1$—appear in the expressions for $\mathbf{T}(0)$ and $\mathbf{M}(0)$, and then only in the terms that are $\mathrm{O}(\varepsilon)$.

(2) From (67),

$$d = (1/2\pi)\mathbf{k}\cdot\mathbf{T}(0), \tag{73}$$

which depends on the axial component of $\mathbf{T}(0)$ only, whereas $\mathbf{b}$, b, and $\mathbf{d}$ involve the elastic constants and the external loads. The expressions for these latter constants simplify considerably if the external loads vanish or for extended orthotropy (all underlined terms vanish). Thus, b and $\mathbf{d}$ are given by

$$2Ab = \mathbf{k}\cdot\mathbf{M}(0) - \mathbf{m}\cdot\mathbf{I}^{-1}\cdot\mathbf{T}(0) + [\pi\mathbf{q} - B\mathbf{m}\cdot(\mathbf{I}^{-1}\times\mathbf{k})]\cdot\underline{\boldsymbol{\gamma}} + \mathrm{O}(\varepsilon) \tag{74}$$

and

$$\pi\mathbf{d} = (\mathbf{I}^{-1}\times\mathbf{k})\cdot\mathbf{M}(0) + \mathbf{k}\times\mathbf{O}(\varepsilon), \tag{75}$$

By the static-geometric duality,

$$2A\underline{\lambda}^* = (\pi\mathbf{q} - B\mathbf{m}\cdot\mathbf{l}^{-1}\times\mathbf{k})\cdot\underline{\gamma}^* + O(\varepsilon), \underline{\boldsymbol{\lambda}}^* = (B/\pi)(\mathbf{l}^{-1}\times\mathbf{k})\cdot\underline{\gamma}^* + \mathbf{k}\times\mathbf{O}(\varepsilon), \tag{76}$$

and

$$\pi\mathbf{d}^* = \mathbf{k}\times\mathbf{O}(\varepsilon). \tag{77}$$

(3) If there are no surface loads and if the three coupling coefficients C^*_{1122} C^*_{1222} and C^*_{1212} vanish, then the $O(\varepsilon)$ and $\mathbf{O}(\boldsymbol{\varepsilon})$ terms in (74) and (75) vanish whereas $\underline{\lambda}^* = 0$ and $\underline{\boldsymbol{\lambda}}^* = \mathbf{d}^* = \mathbf{0}$.

With all the constants in the beamlike solutions now known in terms of $\mathbf{T}(0)$, $\mathbf{M}(0)$, $\mathbf{P}$, and $\mathbf{L}$, we can draw several important conclusions concerning the inextensional part of the displacement, as reflected in the expression (56*) for $\overset{0}{\underline{H}}$.

(1) For a circular cross section, $\overset{0}{\underline{H}} = 0$; [4]
(2) To within a relative error of $O(\varepsilon)$, $\overset{0}{\underline{H}}$ depends on the transverse components of $\mathbf{P}$ only, as (53), $(60)_2$, and (76) show.
(3) These same equations together with $(28)_{2,3}$ show that, unless $A_{1222}C^*_{2222} = \overline{C}_{1222}$, in which case $\overset{0}{\underline{H}} = O(\varepsilon)$, *the presence of surface loads induces a (relatively large), non-rigid-body, inextensional bending displacement.*

To obtain the (dimensionless) beamlike displacement $\mathbf{U} + \varepsilon\mathbf{u}$, we first compute $U = \mathbf{U}\cdot\mathbf{k}$ by setting $\varepsilon = 0$ in $(28)_2$ and using variation of parameters to conclude that

$$\mathscr{M}(U^\bullet) = -\overset{0}{\underline{H}}(y;0) \Rightarrow U^\bullet = \mathbf{t}(y)\cdot\left[\mathbf{C}_1 - \int_0^y \mathbf{n}(\eta)\overset{0}{\underline{H}}(\eta;0)\,\mathrm{d}\eta\right], \tag{78}$$

where $\mathbf{C}_1$ is an unknown constant vector. We next compute $\overset{0}{\underline{H}}$ by inserting (76) into (56*) and noting (59), (62), (63), (68) and (72). Setting $\mathbf{t} = \mathbf{r}^\bullet$ and integrating by parts, we find that

$$\begin{aligned} U &= C_1 + \mathbf{C}_1\cdot\mathbf{r}(y) - \mathbf{r}(y)\cdot\int_0^y \mathbf{n}(\eta)\overset{0}{\underline{H}}(\eta;0)\,\mathrm{d}\eta + \int_0^y \mathbf{r}(\eta)\cdot\mathbf{n}(\eta)\overset{0}{\underline{H}}(\eta;0)\,\mathrm{d}\eta \\ &= C_1 + \mathbf{C}_1\cdot r(y) + \{(\pi/A)[A\mathbf{q}(y) - A(y)\mathbf{q}] + (B/A)[A(y)\mathbf{m} - A\mathbf{m}(y)]\cdot(\mathbf{l}^{-1}\times\mathbf{k}) \\ &\quad + \mathbf{r}(y)\cdot[(1/2)\mathbf{k}\times\mathbf{s}(y)\mathbf{s}(y) - B(y)\mathbf{1} - B\mathbf{k}\times\mathbf{l}(y)\cdot\mathbf{l}^{-1}\times\mathbf{k}]\}\cdot\underline{\gamma}^*, \end{aligned} \tag{79}$$

where $C_1 + \mathbf{C}_1\cdot\mathbf{r}$ is a rigid-body term. [5] As there are no dislocations, $U(2\pi) = U(0)$. Further, because $\mathbf{d}^* = \mathbf{k}\times\mathbf{O}(\varepsilon)$, it follows that $\overset{0}{K}_y = O(\varepsilon)$, which, by $(28)_3$, implies that

$$V = C_2\mathbf{r}(y)\cdot\mathbf{n}(y) + \mathbf{C}_2\cdot\mathbf{t}(y) \tag{80}$$

a rigid-body term. [Recall from (26) that the tangential component of inextensional displacement is $V - xU^\bullet$.]

We may compute $\mathbf{u}$, to within a relative error of $O(\varepsilon)$, by using the strain–stress relations (12) and the strain–displacement relations (27). These calculations are straightforward but tedious in their full generality so we omit them here. However, see (89), (90), (93) and (94) where a restricted form of these relations must be used to compute overall beamlike stress–strain relations.

[4] The "singular" nature of a circular cross section is not surprising if we consider an infinite tube under a constant internal pressure p_0. Since all unknowns in this case are independent of the axial coordinate, (8) and (9) reduce to $N_y^\bullet + \varepsilon\kappa M_y^\bullet = 0$ and $N_y = \rho(p_0 + \varepsilon M_y^{\bullet\bullet})$. Eliminating N_y we obtain $\varepsilon\mathscr{M}(M_y^\bullet) + \rho^\bullet p_0 = 0$. This equation shows that, unless $\rho^\bullet = 0$, i.e., unless the cross section is *circular*, we must take $p_0 = O(\varepsilon)$. In other words, a circular tube is *imperfection sensitive,* even when buckling is not an issue.

[5] The full dimensionless rigid-body displacement has the form $\mathbf{U}_{RB} = \mathbf{D} + \mathbf{R}\times[x\mathbf{k} + \mathbf{r}(y)]$, where $\mathbf{D}$ and $\mathbf{R}$ are constant vectors.

7. Beamlike displacements, rotations, and strains

Let $\mathbf{\Delta}$ and $\mathbf{\Psi}$ denote, respectively, a beamlike virtual displacement and rotation. Taking the dot product of (18) with $\mathbf{\Delta}$, the dot product of (19) with $\mathbf{\Psi}$, and integrating over the length of the shell and by parts to remove derivatives on the force $\mathbf{T}$ and moment $\mathbf{M}$, we arrive at the *identity*

$$(\mathbf{T}\cdot\mathbf{\Delta}+\mathbf{M}\cdot\mathbf{\Psi})_0^l+\int_0^l(\mathbf{P}\cdot\mathbf{\Delta}+\mathbf{L}\cdot\mathbf{\Psi})\,\mathrm{d}x\equiv\int_0^l[\mathbf{T}\cdot(\mathbf{\Delta}'+\mathbf{k}\times\mathbf{\Psi}')+\mathbf{M}\cdot\mathbf{\Psi}']\,\mathrm{d}x \tag{81}$$

The form of the right side of this equation suggests that the virtual extensional and bending strains associated with $\mathbf{T}$ and $\mathbf{M}$ are

$$\mathbf{\Gamma}\equiv\varepsilon^{-1}(\mathbf{\Delta}'+\mathbf{k}\times\mathbf{\Psi}) \tag{82}$$

and

$$\mathbf{\Omega}\equiv\varepsilon^{-1}\mathbf{\Psi}' \tag{83}$$

But if we introduce the *actual* (dimensionless) displacement $\mathbf{U}+\varepsilon\mathbf{u}$, how to define $\mathbf{\Delta}$ and $\mathbf{\Psi}$? The key lies in the term $\mathbf{P}\cdot\mathbf{\Delta}+\mathbf{L}\cdot\mathbf{\Psi}$, although one's first impulse is to look to the first term on the left of (81) as this is the only external work term if there are no surface loads.

In the Sanders–Koiter theory, twice the work done by the surface loads follows from (26) and (31) as

$$2\mathscr{W}=\int_0^l\int_0^{2\pi}\mathbf{p}\cdot(\mathbf{U}+\varepsilon\mathbf{u})\,\mathrm{d}y\,\mathrm{d}x=\int_0^l\int_0^{2\pi}[\overset{0}{p}_x(U+\varepsilon u)+\overset{0}{p}_y(V-xU^{\bullet}+\varepsilon v)]\,\mathrm{d}y\,\mathrm{d}x. \tag{84}$$

(Recall that we have set $\overset{0}{p}=0$.) By (32), (47) and (51),

$$2\mathscr{W}=\int_0^l\int_0^{2\pi}[(\beta+\boldsymbol{\beta}\cdot\mathbf{r})(U+\varepsilon u)+(\alpha-\boldsymbol{\alpha}\cdot\mathbf{s})(V-xU^{\bullet}+\varepsilon v)]\,\mathrm{d}y\,\mathrm{d}x. \tag{85}$$

Further, using (60) and (64) to express (β, $\boldsymbol{\beta}$, α and $\boldsymbol{\alpha}$ in terms of $\mathbf{P}$ and $\mathbf{L}$, we obtain

$$\begin{aligned}2\mathscr{W}=\int_0^l\Bigg\{\mathbf{P}\cdot\int_0^{2\pi}&(1/2\pi)(U+\varepsilon u)\mathbf{k}-(V-xU^{\bullet}+\varepsilon v)(\mathbf{s}/\pi+\mathbf{m}/2A)\cdot\mathbf{I}^{-1}]\,\mathrm{d}y\\&+\mathbf{L}\cdot\int_0^{2\pi}[(1/2A)(V-xU^{\bullet}+\varepsilon v)\mathbf{k}+(1/\pi)(U+\varepsilon u)\mathbf{r}\cdot\mathbf{I}^{-1}\times\mathbf{k}]\,\mathrm{d}y\Bigg\}\,\mathrm{d}x.\end{aligned} \tag{86}$$

Equating the integrand on the right side of this equation to $\mathbf{P}\cdot\mathbf{\Delta}+\mathbf{L}\cdot\mathbf{\Psi}$ and setting $-\int_0^{2\pi}U^{\bullet}\mathbf{s}\,\mathrm{d}y=\int_0^{2\pi}U\mathbf{r}\,\mathrm{d}y$, we arrive at the definitions

$$\begin{aligned}\mathbf{\Delta}&\equiv\int_0^{2\pi}\{(1/2\pi)U\mathbf{k}-[V(\mathbf{s}/\pi+\mathbf{m}/2A)+xU\mathbf{r}/\pi]\cdot^{-1}\}\,\mathrm{d}y+\varepsilon\int_0^{2\pi}[(1/2\pi)u\mathbf{k}-v(\mathbf{s}/\pi+\mathbf{m}/2A)\cdot\mathbf{I}^{-1}]\,\mathrm{d}y\\&\equiv\overset{0}{\mathbf{W}}+x\overset{1}{\mathbf{W}}+\varepsilon\mathbf{w}(x;\varepsilon)\end{aligned} \tag{87}$$

and

$$\begin{aligned}\mathbf{\Psi}&\equiv\int_0^{2\pi}[(1/2A)V\mathbf{k}+(1/\pi)U\mathbf{r}\cdot\mathbf{I}^{-1}\times\mathbf{k}]\,\mathrm{d}y\quad+\varepsilon\int_0^{2\pi}[(1/2A)v\mathbf{k}+(1/\pi)u\mathbf{r}\cdot\mathbf{I}^{-1}\times\mathbf{k}]\,\mathrm{d}y\\&\equiv\overset{0}{\mathbf{\Phi}}+\varepsilon\boldsymbol{\phi}(x;\varepsilon)\end{aligned} \tag{88}$$

where $\overset{0}{\mathbf{W}}$, $\overset{1}{\mathbf{W}}$, and $\overset{0}{\mathbf{\Phi}}$ are unknown constants. Note that the definitions of $\mathbf{\Delta}$ and $\mathbf{\Psi}$ are *strictly kinematic, depending on neither material properties nor loads*. [We should also point out that $\mathbf{\Delta}$ and $\mathbf{\Psi}$ do *not* correspond to what Ladevèze and Simmonds (1998) call *generalized displacements*, although both represent measures of overall, beamlike displacements and rotations.]

Expressions for the beamlike strains follow from (27), (82), (83), (87) and (88), and the relations, $\int_0^{2\pi} u^\bullet \mathbf{s}\,\mathrm{d}y = -\int_0^{2\pi} u\mathbf{r}\,\mathrm{d}y$ and $\int_0^{2\pi} u^\bullet\,\mathrm{d}y = 0$ as

$$\boldsymbol{\Gamma} = \int_0^{2\pi} [(1/2\pi)E_x\mathbf{k} - 2E(\mathbf{s}/\pi + \mathbf{m}/2A)\cdot \mathbf{I}^{-1}]\,\mathrm{d}y \tag{89}$$

and

$$\boldsymbol{\Omega} = \int_0^{2\pi} [(1/A)E\mathbf{k} + (1/\pi)E_x\mathbf{r}\cdot\mathbf{I}^{-1}\times\mathbf{k}]\,\mathrm{d}y \tag{90}$$

To further reduce the work identity (81), note that it must hold, in particular, if $\varepsilon = 0$, in which case, by (82), (83), (87) and (88),

$$[\mathbf{T}\cdot(\overset{0}{\mathbf{W}} + x\overset{\mathbf{1}}{\mathbf{W}}) + \mathbf{M}\cdot\overset{0}{\boldsymbol{\Phi}}]_0^l + l\{\mathbf{P}\cdot[\overset{0}{\mathbf{W}} + (1/2)l\overset{\mathbf{1}}{\mathbf{W}}] + \mathbf{L}\cdot\overset{0}{\boldsymbol{\Phi}}\} = 0. \tag{91}$$

That is, *the external work done in any beamlike inextensional deformation vanishes*. But this is obvious physically because, in the last lines of (87) and (88), the terms $\overset{0}{\mathbf{W}} + x\overset{1}{\mathbf{W}} = \overset{0}{\mathbf{W}} + x\overset{0}{\boldsymbol{\Phi}}\times\mathbf{k}$ and $\overset{0}{\boldsymbol{\Phi}}$ are merely beamlike rigid-body displacements.

It now follows from (82), (83), (87) and (88) that (81) reduces to

$$(\mathbf{T}\cdot\mathbf{w} + \mathbf{M}\cdot\boldsymbol{\phi})_0^l + \int_0^l (\mathbf{P}\cdot\mathbf{w} + \mathbf{L}\cdot\boldsymbol{\phi})\,\mathrm{d}x \equiv \int_0^l (\mathbf{T}\cdot\boldsymbol{\Gamma} + \mathbf{M}\cdot\boldsymbol{\Omega})\,\mathrm{d}x \tag{92}$$

Our final task in this section is to obtain stress–strain relations for the beamlike solutions. Noting that taking $\overset{0}{p} = 0$ and $\mathbf{d}^* = \mathbf{e}^* = \mathbf{0}$ implies that $\overset{\mathbf{0}}{N}_y$ and $\overset{\mathbf{0}}{K}_y$ are $\mathrm{O}(\varepsilon)$ and neglecting terms of relative $\mathrm{O}(\varepsilon)$, we have, from (10), (12), (16), (34)–(36*), (89) and (90),

$$\boldsymbol{\Gamma} = \int_0^{2\pi} [(1/2\pi)(-2\underline{A_{2212}}G + N_x - 2\underline{C_{2212}}\underline{H})\mathbf{k} + 2(-2A_{1212}G + \underline{A_{1222}}N_x - 2C_{1212}\underline{H})(\mathbf{s}/\pi + \mathbf{m}/2A)\cdot\mathbf{I}^{-1}]\,\mathrm{d}y \tag{93}$$

$$\boldsymbol{\Omega} = \int_0^{2\pi} [(1/A)2A_{1212}G - \underline{A_{1222}}N_x + C_{1212}\underline{H})\mathbf{k} + (1/\pi)(-2\underline{A_{2212}}G + N_x - 2\underline{C_{2212}}\underline{H})\mathbf{k}\times\mathbf{I}^{-1}\cdot\mathbf{r}]\,\mathrm{d}y \tag{94}$$

For simplicity, we now assume that $A_{1211} = A_{1222} = \overline{C}^*_{1222} = C^*_{1222} = 0$, i.e., we drop all underlined terms. (Recall that we have called these conditions *extended orthotropy*.)

Then, with the neglect of further terms of $\mathrm{O}(\varepsilon)$, (93) and (94) reduce to

$$\boldsymbol{\Gamma} = \int_0^{2\pi} [(1/2\pi)N_x\mathbf{k} - 4A_{1212}G(\mathbf{s}/\pi + \mathbf{m}/2A)\cdot\mathbf{I}^{-1}]\,\mathrm{d}y \tag{95}$$

and

$$\boldsymbol{\Omega} = \int_0^{2\pi} [(2/A)A_{1212}G\mathbf{k} + (1/\pi)N_x\mathbf{r}\cdot\mathbf{I}^{-1}\times\mathbf{k}]\,\mathrm{d}y \tag{96}$$

Two of the integrals in (95) and (96) follow directly from (15), (20) and (22), namely,

$$\int_0^{2\pi} N_x(x,y;\varepsilon)\,\mathrm{d}y = \mathbf{k}\cdot\mathbf{T}(x) \tag{97}$$

and

$$\int_0^{2\pi} N_x(x,y;0)\mathbf{r}(y)\,\mathrm{d}y = \mathbf{k}\times\mathbf{M}(x). \tag{98}$$

For the others involving G, we first use (35), (49) and (56). Because the underlined term in (56) vanishes, we find that

$$\int_0^{2\pi} G(\mathbf{s}/\pi + \mathbf{m}/2A)\,\mathrm{d}y = -\mathbf{b}\cdot\mathbf{J} + (\pi/A)b\mathbf{m} + x[\boldsymbol{\alpha}\cdot\mathbf{J} - (\pi/A)\alpha\mathbf{m}] \tag{99}$$

and

$$\int_0^{2\pi} G\,\mathrm{d}y = 2\pi(b - x\alpha), \tag{100}$$

where

$$\mathbf{J}(y) = (1/\pi)\int_0^{y} \mathbf{s}(\eta)\mathbf{s}(\eta)\,\mathrm{d}\eta = \mathbf{J}^{\mathrm{T}}(y), \mathbf{J} = \mathbf{J}(2\pi). \tag{101}$$

(For a circle, $\mathbf{J} = 1$.)

Next, we use $(60)_2$, $(64)_1$, $(69)_2$, and (73)—with the underlined terms neglected—to express $\boldsymbol{\alpha}$, α, $\mathbf{b}$, and b in terms of $\mathbf{P}$, $\mathbf{L}$, $\mathbf{T}(0)$, and $\mathbf{M}(0)$. Noting that (24), (25) and (31) imply that, for external loads constant along the axis of the tube,

$$\mathbf{T} = \mathbf{T}(0) - x\mathbf{P} \tag{102}$$

and

$$\mathbf{M} = \mathbf{M}(0) - x[\mathbf{k}\times\mathbf{T}(0) + \mathbf{L}] + (x^2/2)\mathbf{k}\times\mathbf{P}, \tag{103}$$

we find altogether that (95) and (96) take the form

$$\boldsymbol{\Gamma} = \mathbf{w}' + \mathbf{k}\times\boldsymbol{\phi} = \boldsymbol{\Lambda}_{\mathbf{T}}\cdot\mathbf{T} + \boldsymbol{\Lambda}\cdot\mathbf{M} \tag{104}$$

and

$$\boldsymbol{\Omega} = \boldsymbol{\phi}' = \boldsymbol{\Lambda}^{\mathrm{T}}\cdot\mathbf{T} + \boldsymbol{\Lambda}_{\mathbf{M}}\cdot\mathbf{M}, \tag{105}$$

where

$$\boldsymbol{\Lambda}_{\mathbf{T}} = (1/2\pi)\mathbf{kk} + 4A_{1212}\mathbf{I}^{-1}\cdot[(1/\pi)\mathbf{J} + (\pi/2A^2)\mathbf{mm}]\cdot\mathbf{I}^{-1} \tag{106}$$

$$\boldsymbol{\Lambda} = -(2\pi/A^2)A_{1212}\mathbf{I}^{-1}\cdot\mathbf{mk} \tag{107}$$

$$\boldsymbol{\Lambda}_{\mathbf{M}} = (2\pi/A^2)A_{1212}\mathbf{kk} - (1/\pi)(\mathbf{k}\times\mathbf{I}^{-1}\times\mathbf{k}) \tag{108}$$

the superscript “T” denoting “transpose”. For a circle,

$$\boldsymbol{\Lambda}_T = (1/2\pi)\mathbf{kk} + (4/\pi)A_{1212}\mathbf{1}, \boldsymbol{\Lambda} = \mathbf{0}, \boldsymbol{\Lambda}_{\mathbf{M}} = (1/\pi)(2A_{1212}\mathbf{kk} + \mathbf{1}) \tag{109}$$

Further, if the tube is isotropic, $2A_{1212} = 1 + v$, where v is Poisson's ratio, so that $\boldsymbol{\Lambda}_T = (1/2\pi)[\mathbf{kk} + 4(1+v)\mathbf{1}]$ and $\boldsymbol{\Lambda}_M = (1/\pi)[(1+v)\mathbf{kk} + \mathbf{1}]$. These expressions agree with the analytical expressions from three-dimensional elasticity given in equation (4.29) of Sanchez (2001) for a tube of inner radius a and outer radius b in the limit as $a \to b$.

The beamlike stress–strain relations (104) and (105) have the same form as in equation (6) of Ladevèze and Simmonds (1998), except that, as mentioned earlier, here the beamlike displacements and rotations—

and hence the strains—are different and there is no explicit dependence on the distributed external loads. [However, such load terms do appear if the underlined terms in (93) and (94) are not zero.]

The solution of the beamlike equations is now straightforward: with $\mathbf{T}$ and $\mathbf{M}$ given explicitly by (102) and (103), we first integrate (105) to obtain $\boldsymbol{\phi}$ and then (104) to obtain $\mathbf{w}$. If we suppress rigid-body terms, then the boundary conditions at $x = 0, l$, be they kinetic, kinematic, or a mixture, determine the unknowns $\mathbf{T}(0)$ and $\mathbf{M}(0)$.

8. Example: an orthotropic elliptical tube

Let $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ denote the standard set of orthonormal base vectors in a fixed Cartesian reference frame and let

$$R\mathbf{r} = a\cos\phi\mathbf{i} + b\sin\phi\mathbf{j}, \quad 0 \leqslant \phi < 2\pi, \tag{110}$$

where $\pi R/2a = E(\sqrt{1 - b^2/a^2})$, the complete elliptic integral of the second kind. The dimensionless distance y around the tube is related to ϕ by the initial value problem

$$y'(\phi) = (a/R)\sqrt{\sin^2\phi + (b/a)^2\cos^2\phi} \equiv \tau(\phi; b/a) = \sum_0^\infty \tau_n \cos 2n\phi, \quad 0 < \phi \leqslant 2\pi, \quad y(0) = 0 \tag{111}$$

By either using (111) or appealing to symmetry arguments, we conclude that for an elliptic cross section,

$$\mathbf{m} = \mathbf{0}, \quad \mathbf{I} = I_{11}\mathbf{ii} + I_{22}\mathbf{jj}, \quad \mathbf{J} = J_{11}\mathbf{ii} + J_{22}\mathbf{jj} \tag{112}$$

Although it is possible to express the components of $\mathbf{I}$ and $\mathbf{J}$ in terms of the Fourier coefficients τ_n in (111), it is much simpler to compute them numerically, using standard routines for integrating first-order differential equations with initial conditions. This may be done in two steps.

First, we need to compute $\mathbf{s}(0) = (l/2\pi)\int_0^{2\pi} y\mathbf{r}\,\mathrm{d}y \equiv c_2(2\pi)\mathbf{j}$, where, by (46) and symmetry, $\mathbf{i}\cdot\mathbf{s}(0) = 0$. Then using (48), (110) and (111) and with $(\ )' = d(\ )/d\phi$, we solve numerically

$$\underline{X}' \equiv \begin{bmatrix} y \\ c_2 \end{bmatrix}' = \tau(\phi; b/a)\begin{bmatrix} 1 \\ (1/2\pi)(b/R)y\sin\phi \end{bmatrix}, \quad 0 < \phi \leqslant 2\pi, \tag{113}$$

subject to the initial condition $\underline{X}(0) = \underline{0}$. Next, noting (48), (59) and (101), and with

$$\mathbf{s} = s_1(\phi)\mathbf{i} + s_2(\phi)\mathbf{j}, \tag{114}$$

we solve

$$\underline{Y}' \equiv \begin{bmatrix} I_{11} \\ I_{22} \\ J_{11} \\ J_{22} \\ s_1 \\ s_2 \end{bmatrix} = \tau(\phi; b/a)\begin{bmatrix} (1/\pi)(a/R)^2\cos^2\phi \\ (1/\pi)(b/R)^2\sin^2\phi \\ (1/\pi)s_1^2(\phi) \\ (1/\pi)s_2^2(\phi) \\ (a/R)\cos\phi \\ (b/R)\sin\phi \end{bmatrix}, \quad 0 < \pi \leqslant 2\pi, \tag{115}$$

subject to the initial condition $\underline{Y}(0) = [0, 0, 0, 0, 0, c_2(2\pi)]^{\mathrm{T}}$, the last component of $\underline{Y}(0)$ coming from the solution of (113) evaluated at $\phi = 2\pi$. This yields

$$\{\mathbf{I}, \mathbf{J}\} = \{I_{11}, J_{11}\}\mathbf{ii} + \{I_{11}, J_{22}\}\mathbf{jj} \quad \text{at } \phi = 2\pi \tag{116}$$

which allows us to compute the tensor coefficients $\boldsymbol{\Lambda}_{\mathbf{T}}$ and $\boldsymbol{\Lambda}_{\mathbf{M}}$ which appear in the beamlike strain–stress relations (104) and (105).

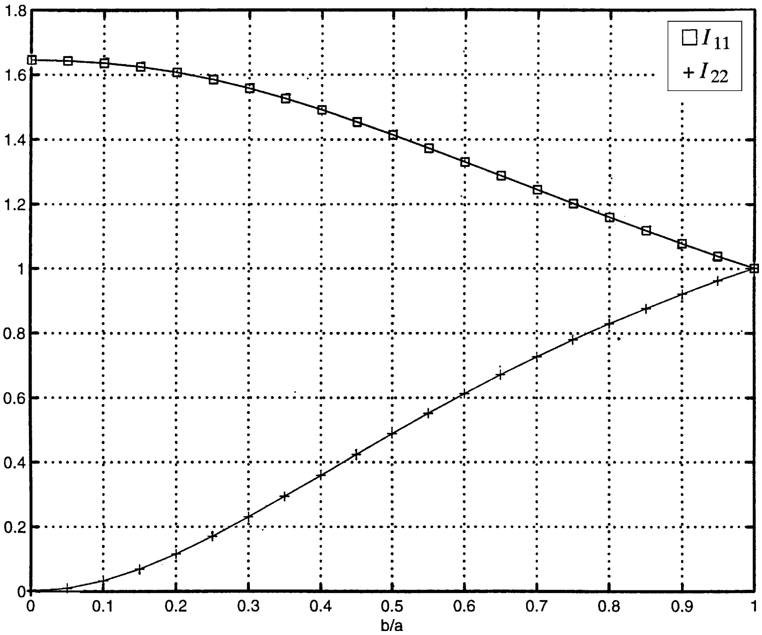

Fig. 1. Non-zero components of the tensor $\mathbf{I} \equiv \mathbf{I}(2\pi)$ defined in (59) for elliptical cross sections with various ratios of semi-minor to semi-major axes (b/a).

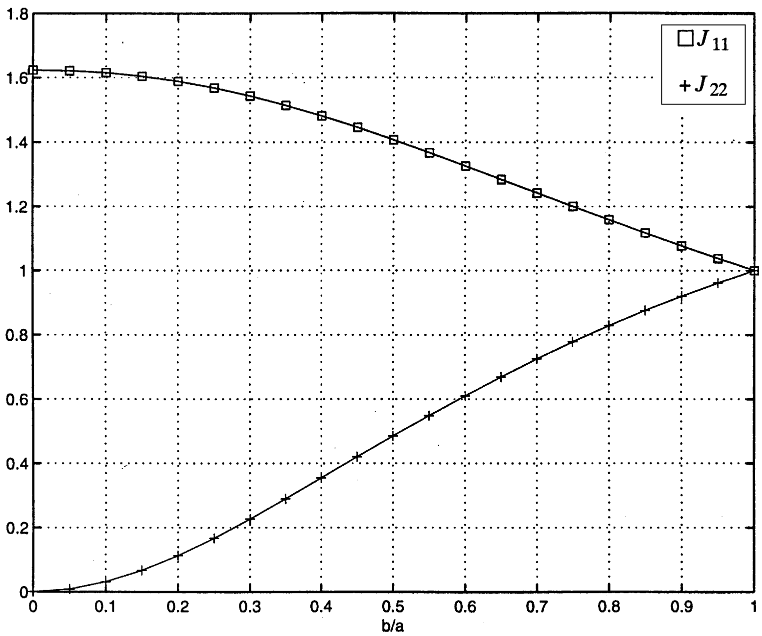

Fig. 2. Non-zero components of the tensor $\mathbf{J} \equiv \mathbf{I}(2\pi)$ defined in (101) for eliptical cross sections with various ratios of semi-minor to semi-major axes (b/a).

We have solved these equations for $b/a \in [0,1]$. The resulting values for the components of $\mathbf{I}$ and $\mathbf{J}$ are displayed in Figs. 1 and 2. For a circle $(b/a = 1)$, $I_{11} = I_{22} = J_{11} = J_{22} = 1$; for a flattened ellipse $(b/a = 0)$, $I_{11} = \pi^2/6, I_{22} = 0, J_{11} = \pi^4/60, J_{22} = 0$. Values for the components of $\boldsymbol{\Lambda}_\mathbf{T}$ and $\boldsymbol{\Lambda}_\mathbf{M}$ follow readily from (106) and (108), where the area of the cross section $\mathscr{S}$ is given by $A = (b/a)\pi^3/4E^2(\sqrt{1 - b^2/a^2})$. Even for relatively thick tubes $(h/b = 0.1)$, these values for the components of $\boldsymbol{\Lambda}_T$ and $\boldsymbol{\Lambda}_M$ are virtually indistinguishable from those Sanchez (2001) computed—necessarily more expensively—from three-dimensional elasticity.

Appendix A

Herein, we show that, to within a relative error of $O(\varepsilon)$, the effective edge stress resultant $\mathbf{N}_x$, defined in (15), can be cast into a form analogous to the Saint–Venant normal traction, $\sigma_{sv}N|s_x$ given by equation (7) of Ladevèze and Simmonds (1998).

Starting from (15) and using, successively, (35), (47), (56), $(60)_2$, $(64)_1$, (67), (69), (73), (74), (97), (102), and (103), we find that

$$\mathbf{N}_x = \overset{0}{\mathbf{A}}(y;\varepsilon) \cdot \mathbf{T}(x) + \overset{0}{\mathbf{B}}(y;\varepsilon) \cdot \mathbf{M}(x) + (1/\pi)\underline{\Gamma}[\mathbf{I}^{-1} \cdot \mathbf{g}(y) \cdot \mathbf{P}]\mathbf{t}(y) + \mathbf{O}(\varepsilon), \tag{A.1}$$

where

$$\overset{0}{\mathbf{A}}(y;0) = (1/2\pi)\mathbf{kk} - \mathbf{t}(y)[(1/2A)\mathbf{m} + (1/\pi)\mathbf{s}(y)] \cdot \mathbf{I}^{-1} \tag{A.2}$$

$$\overset{0}{\mathbf{B}}(y;0) = (1/2A)\mathbf{t}(y)\mathbf{k} + (1/\pi)\mathbf{kr}(y) \cdot (\mathbf{I}^{-1} \times \mathbf{k}) \tag{A.3}$$

$$\mathbf{g} = (\pi/2A)\mathbf{q} - B[(1/2A)\mathbf{m} + (1/\pi)\mathbf{s}(y)] \cdot \mathbf{I}^{-1} \times \mathbf{k} + \mathbf{v}(y), \tag{A.4}$$

$\underline{\Gamma}$ is a material constant given by (53), and $\mathbf{v}$(y) is a known function given by (54). Note that, to $O(\varepsilon)$, material properties enter (A.1) only through the scalar factor $\underline{\Gamma}$ (which vanishes for extended orthotropy, as we have noted earlier).

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